Quadratic Transformation of the Exponential–Gamma Distribution for Flexible Lifetime Modelling

Taiwo M. Ayeni[1] Elijah A. Odukoya[2] Yichuan Zhao[1]

[1]Department of Mathematics & Statistics, Georgia State University, Atlanta, GA [2]Federal University of Technology & Environmental Sciences, Iyin-Ekiti, Nigeria

Abstract

This study introduces the Quadratic-Transformed Exponential–Gamma (QTEG) distribution for modelling right-skewed lifetime data with non-constant hazard behaviour. The model is obtained by applying the quadratic transformation $Y = X^2$ to a corrected exponential–gamma baseline, which reduces to a Gamma distribution after normalisation. QTEG provides closed-form reliability functions, a semi-closed MLE, and flexible hazard shapes. Simulation and real-data results show stable estimation and improved fit over common lifetime models.

Main Contributions

- ✓ Derives the EGD normalising constant; quantifies its effect on likelihood-based inference.
- ✓ Develops the QTEG density, CDF, survival, hazard, entropy, quantile, and order-statistic functions in closed form.
- ✓ Provides a semi-closed MLE ($\hat{\beta}$ exact given α; α via profile likelihood / L-BFGS-B) with analytic Fisher information and proof of estimator uniqueness.
- ✓ Monte Carlo study across 3 hazard regimes, 4 sample sizes, $N = 500$ replications: 100% convergence and well-calibrated coverage.
- ✓ Strong fit on three benchmark lifetime datasets against eight competing distributions on seven criteria.

Motivation and Gap

Lifetime data often show early failure, random failure, or wear-out patterns. Classical two-parameter models may be too restrictive. All three benchmark datasets have CV> 1, skewness> 2, and excess kurtosis> 4 — exactly the regime where the QTEG tail $e^{-\beta\sqrt{y}}$ outperforms classical families.

Model	Main limitation
Exponential	Constant hazard only
Gamma / Weibull	Monotone hazard shapes only
Log-normal	Flexible tail but restricted hazard structure

Need: a tractable two-parameter model capturing decreasing, approximately flat, and unimodal hazard behaviours, and statistically distinct from the Gamma (tail $e^{-\beta\sqrt{y}}$ is strictly heavier than $e^{-\beta y}$; no reparametrisation connects them).

Core Idea

$$Y = X^2, \quad X \sim \mathrm{Gamma}(\alpha, \beta)$$

The quadratic transformation replaces $e^{-\beta x}$ with the strictly heavier decay $e^{-\beta\sqrt{y}}$, enriching hazard shapes within a two-parameter structure that remains mathematically tractable.

Transformation and Final QTEG Density

For $Y = X^2$, $x = \sqrt{y}$ and $|d\sqrt{y}/dy| = (2\sqrt{y})^{-1}$:

$$f_Y(y) = \frac{\beta^\alpha}{2\Gamma(\alpha)} y^{(\alpha-2)/2} e^{-\beta\sqrt{y}}, \qquad y > 0$$

- α governs hazard shape (density unbounded at origin when $\alpha < 2$; remains integrable).
- β is a pure scale parameter (CV is independent of β).

Selected References

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Code & Data

github.com/ayeni-T/QTEG-distribution

Scan to access Python code and datasets

Model Construction Pipeline

1. **Raw EGD construction** $g(x) = \dfrac{\lambda^{k+1}}{\Gamma(k)} x^{k-1} e^{-2\lambda x}$ not normalised: integrates to $\lambda/2^k \neq 1$

↓

2. **Normalisation** $C = 2^k/\lambda$ gives a valid density. *Omitting C shifts all ICs by $|2n\log(\lambda/2^k)|$ — invalidating model comparisons.*

↓

3. **Reparametrisation** $\alpha = k,\ \beta = 2\lambda$ gives $f_X(x) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ $[\equiv \mathrm{Gamma}(\alpha, \beta)]$

↓

4. **Quadratic transformation** $Y = X^2$ gives QTEG with tail decay $e^{-\beta\sqrt{y}}$

Theoretical Shape and Hazard Behaviour

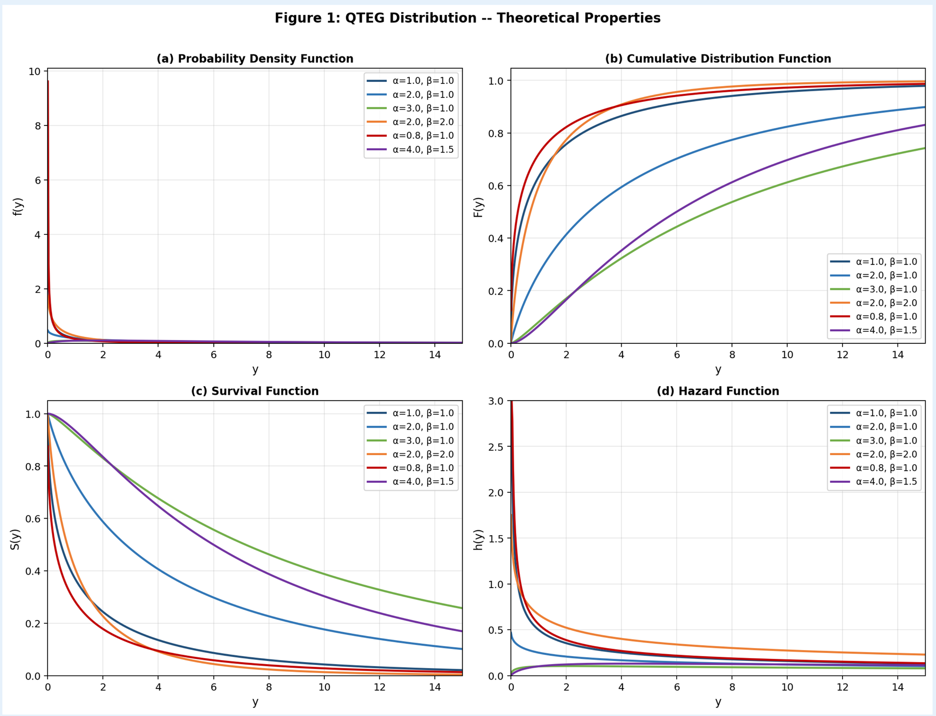

Fig. 1: PDF, CDF, survival, and hazard functions for selected (α, β). Hazard panel shows the three regimes governed by α.

$\alpha < 2$	$\alpha \approx 2$	$\alpha > 2$
Decreasing hazard	Approximately flat	Unimodal / wear-out

"Approximately flat" applies for $\alpha \in [1.8, 2.2]$; at $\alpha = 2$ the polynomial factor $y^{(\alpha-2)/2} = 1$ is exactly flat.

Estimation and Simulation Evidence

Score equation for β yields a semi-closed estimator; α is found by one-dimensional profile-likelihood root search (L-BFGS-B, 8-point grid):

$$\widehat{\beta}(\alpha) = \frac{\alpha}{\overline{\sqrt{y}}}, \qquad \overline{\sqrt{y}} = \frac{1}{n}\sum_{i=1}^{n} \sqrt{y_i}$$

Simulation design: $N = 500$ replications; $n \in \{30, 50, 100, 200\}$; three scenarios spanning all hazard regimes — Sc. 1 ($\alpha = 1.5, \beta = 0.5$), Sc. 2 ($\alpha = 2.0, \beta = 1.0$), Sc. 3 ($\alpha = 3.0, \beta = 2.0$).

100%	**93.8–96.4%**	**↓ O(1/n)**
Convergence	Empirical CP	Bias & RMSE
all 12 scenario–n cells	nominal 95%	as n increases

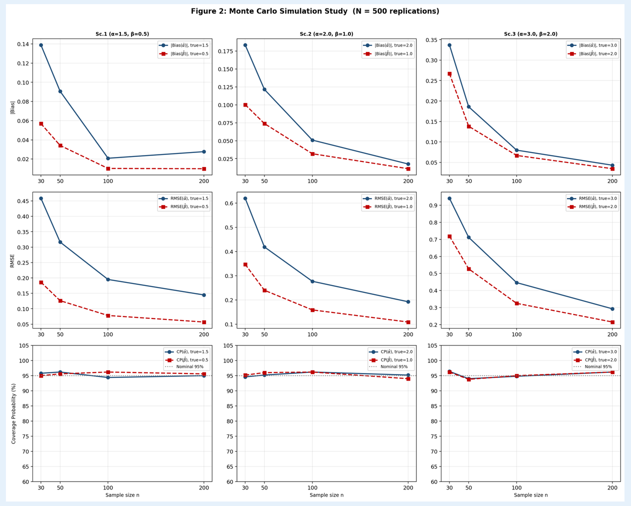

Fig. 2: Monte Carlo diagnostics ($N = 500$) — bias (top), RMSE (middle), and empirical 95% coverage probability (bottom; dashed = nominal) vs. n for Scenarios 1–3.

Model Comparison — Bladder Cancer (DS1, $n = 128$)

All fits by MLE; bold = best per criterion. QTEG is the only model with KS $p > 0.05$, confirming adequate fit at conventional significance.

Model	LogLik	AIC	BIC	CAIC	HQIC	KS p
QTEG	**−410.86**	**825.72**	**831.42**	**825.81**	**828.03**	**0.953**
EG/Gamma	−413.37	830.73	836.43	830.83	833.05	0.484
Weibull	−414.08	832.16	837.87	832.26	834.48	0.543
Exp-Gamma	−411.34	828.68	837.24	828.88	832.16	0.807
Kum-Gamma	−411.14	830.28	841.69	830.60	834.91	0.869
Lognormal	−415.09	834.18	839.89	834.28	836.50	0.698
Exponential	−414.33	830.67	833.52	830.70	831.83	0.329

See Fig. 3 for fitted densities across all three datasets.

Real Data Applications — Summary

Dataset	n	Skew	Ex. Kurt.	KS p
Bladder cancer (DS1)	128	3.29	15.48	**0.953**
Boeing failures (DS2)	213	2.11	4.93	**0.904**
Melanoma survival (DS3)	205	2.17	5.44	**0.587**

High skewness and excess kurtosis signal heavier tails than classical models accommodate — exactly the QTEG advantage. QTEG achieved best AIC, BIC, CAIC, and HQIC across all three datasets. Kum-Gamma (4 parameters) does not outperform QTEG on BIC or CAIC in any dataset.

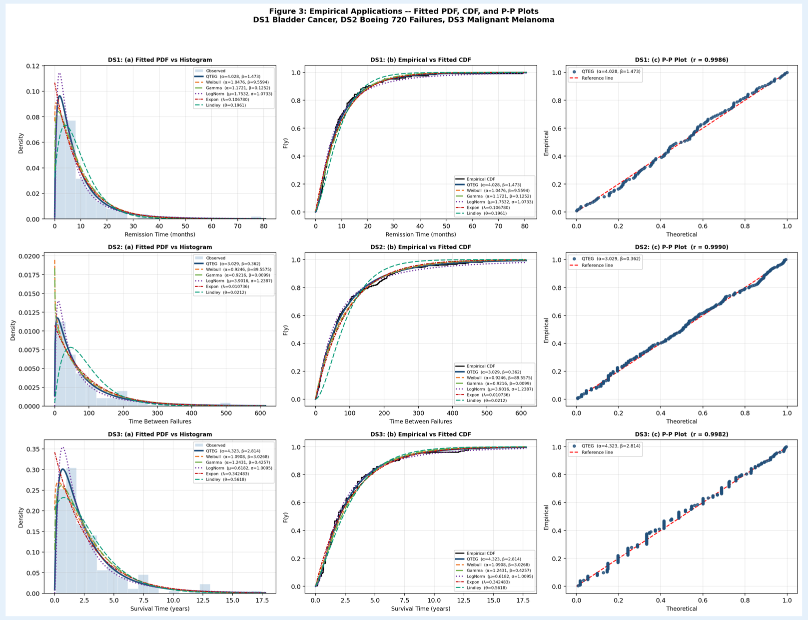

Fig. 3: Fitted density vs. histogram (a), empirical vs. fitted CDF (b), and P-P plot (c) for DS1 (top), DS2 (middle), DS3 (bottom).

Fitted Hazard Functions

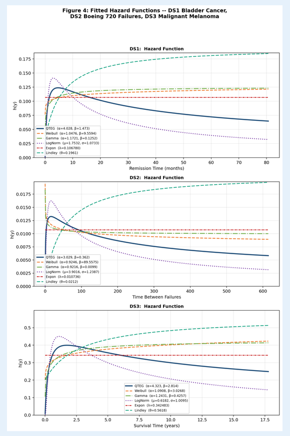

Fig. 4: Fitted hazards for DS1 ($\hat{\alpha} = 4.03$, unimodal), DS2 ($\hat{\alpha} = 3.03$, declining from early peak), and DS3 ($\hat{\alpha} = 4.32$, unimodal) — consistent with theoretical hazard regimes.

Conclusion

QTEG is a tractable two-parameter alternative to classical lifetime models, statistically distinct from the Gamma via its strictly heavier tail $e^{-\beta\sqrt{y}}$. The MLE achieves 100% convergence, monotone consistency, accurate Hessian standard errors, and empirical coverage of 93.8–96.4% across all simulated scenarios. QTEG outperforms all eight competing models on all four information criteria across three benchmark datasets while remaining parsimonious.

Future Work

- Bayesian estimation (β admits a conjugate Gamma posterior).
- Censored-data inference (Type-I, II, progressive) and survival regression.
- Jackknife empirical likelihood goodness-of-fit testing for QTEG.
- Generalised transformation family $Y = X^q$, $q > 0$.

Contact: tayeni2@gsu.edu — Code & Data: github.com/ayeni-T/QTEG-distribution — MSC: 62F10 · 62E15 · 62N05 · 60E05

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