The Quadratic-Transformed Exponential-Gamma (QTEG) Distribution and Its Empirical Likelihood Extension for Survival Modelling

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End-of-Year Research Event | April 10, 2026

The Challenge

Lifetime data in reliability and biomedical studies exhibit:

- Heavy right tails
- Non-monotone (unimodal) hazard rates

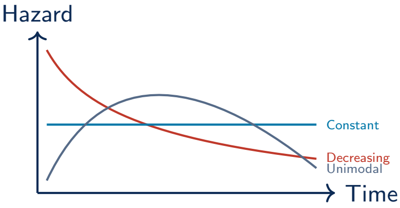
Limitations of Classical Models

[Nadarajah and Kotz, 2006, Almalki and Nadarajah, 2014]

Model	Limitation
Exponential	Only constant hazard
Gamma / Weibull	Monotone hazard only
Multi-parameter	Loses tractability

The Gap

A **flexible** yet **tractable** two-parameter lifetime model that accommodates all hazard regimes within a single parsimonious family



The EGD baseline [Ogunwale et al., 2019]

The Exponential-Gamma Distribution (EGD) is formed by multiplying Exponential and Gamma densities. The resulting raw density is **not normalised**:

$$\int_0^\infty f_{\text{raw}}(x)\,dx = \frac{\lambda}{2^k} \neq 1$$

After normalisation [Casella and Berger, 2002]

Applying the normalising constant $C = 2^k/\lambda$ and reparametrising $\alpha = k$, $\beta = 2\lambda$:

$$f_X(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\,x^{\alpha-1}\,e^{-\beta x} \;\equiv\; \text{Gamma}(\alpha,\beta)$$

Three-point summary

- Raw EGD is **not a valid density**
- After normalisation $\rightarrow$ **Gamma**(α, β)
- Ensures **valid likelihood inference** and direct AIC/BIC comparability

"The correction ensures that all inference is built on a rigorous probabilistic foundation."

Functional Variable Transformation (FVT)

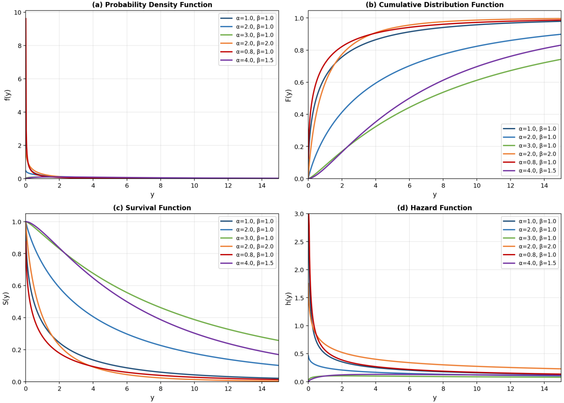
[Alzaatreh et al., 2013, Casella and Berger, 2002]

For $Y = g(X)$ with g strictly increasing:

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\frac{d}{dy}g^{-1}(y)\right|$$

Transform	$g(x)$	$g^{-1}(y)$	Tail of f_Y
Quadratic	x^2	$\sqrt{y}$	$e^{-\beta\sqrt{y}}$
Identity	x	y	$e^{-\beta y}$
Power (q)	x^q	$y^{1/q}$	$e^{-\beta y^{1/q}}$

Gamma(α, β) **baseline** X → $Y = X^2$, $g^{-1} = \sqrt{y}$ → **QTEG**(α, β) **tail** $e^{-\beta\sqrt{y}}$



PDF, CDF, survival function, and hazard of QTEG

Three modelling gains:

- Tail: $e^{-\beta x} \to e^{-\beta\sqrt{y}}$ (heavier)
- Variance amplified: no extra parameters
- Hazard: decreasing / constant / unimodal

The QTEG Model: Density, Hazard, and MLE

Probability Density Function

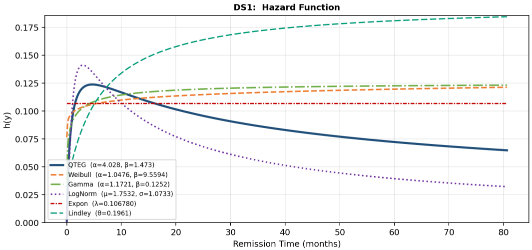
$$f_Y(y;\alpha,\beta) = \frac{\beta^\alpha}{2\,\Gamma(\alpha)}\, y^{(\alpha-2)/2}\, e^{-\beta\sqrt{y}}, \quad y > 0$$

Hazard Shape: controlled by α

α	Regime	Application
< 2	Decreasing	Infant mortality
≈ 2	Constant	Random failure
> 2	Unimodal	Wear-out

Semi-closed MLE

Score equations yield $\hat{\beta}(\alpha) = \alpha/\bar{y}^{1/2}$, so estimation reduces to solving a single equation in α. The Fisher information matrix is positive-definite, guaranteeing a unique MLE.

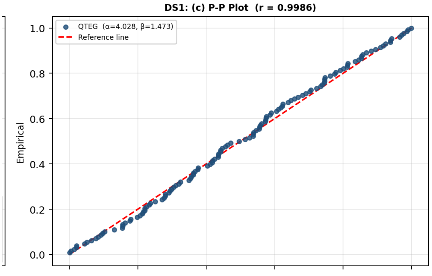


QTEG outperforms 8 competing models

Dataset	QTEG AIC	Next-best
DS1 Bladder (n=128)	**825.7**	828.7
DS2 Boeing (n=213)	**2353.1**	2359.1
DS3 Melanoma (n=205)	**839.1**	843.1

Beats Weibull, Gamma, LogNorm, Exponential, EGD, Exp-Gamma, Kum-Gamma, Lindley on all 4 criteria.

Empirical Fit: P-P Plot (DS1 Bladder Cancer)



Dataset	KS p-val	P-P r
DS1 Bladder (n=128)	0.953	0.9986
DS2 Boeing (n=213)	0.904	0.9990
DS3 Melanoma (n=205)	0.587	0.9982

KS $p > 0.05$ and P-P $r > 0.998$ for all three datasets.

QTEG provides an excellent fit for heavily right-skewed lifetime data.

Monte Carlo Performance

- **100% convergence** across all 12 scenario–n combinations
- Bias and RMSE **decrease monotonically** with n in all scenarios
- Empirical coverage probability: **93.8%–96.4%** (nominal 95%)
- AvgSE closely tracks empirical RMSE — Hessian approximation is accurate

QTEG MLE is **consistent, convergent,** and **correctly calibrated** at all sample sizes.

Extension: Jackknife Empirical Likelihood Inference

① Problem

Wald CIs assume normality of $\hat{\alpha}$. For QTEG at small n:

- Sampling dist. is **right-skewed**
- Wald **under-covers**
- CP $\approx 85\%$ at $n = 30$

② Idea: Use EL

Empirical Likelihood avoids normality:

- **No normality** assumption
- Wilks theorem gives $\ell_{\text{JEL}} \xrightarrow{d} \chi_1^2$
- Range-respecting CI

Construct pseudo-values, apply EL, invert the ratio.

③ Contribution

JEL-based CIs for QTEG:

- Parameters α, β
- Survival $S(t_0; \alpha, \beta)$
- Wilks theorems: JEL, AJEL, EJEL

[Zhao et al., 2015] (CSDA)

"Construct pseudo-values and apply empirical likelihood to obtain confidence intervals that respect the data's true shape."

Summary

Proposed Framework: QTEG Model

- Flexible 2-parameter lifetime model via $Y = X^2$
- Decreasing, constant, or unimodal hazard
- Tractable MLE and Fisher information
- Outperforms 8 models on 3 benchmark datasets

Status: Under review, *Statistical Papers*

Inference Extension: JEL

- Jackknife EL CIs for α, β, and $S(t_0)$
- Wilks theorems: JEL, AJEL, EJEL
- Bivariate joint confidence region
- Monte Carlo $N = 5{,}000$ in progress

Status: Working paper, draft complete

Future: GoF test via characterisation-based U-statistic JEL | Extension to right-censored data

Thank you | Questions welcome

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